Solving punctured multi black hole initial data with finite element method

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Puncture type initial data in NR

\[ R + K^2 - K_{ij}K^{ij} = 0 \]
\[ \nabla_j K^j_i - \nabla_i K = 0 \]

\[ \gamma_{ij} = \psi^4 f_{ij}, \]
\[ K_{ij} = \psi^{-2} \hat{K}_{ij}, \]

\[ \hat{K}_{ij} = \frac{3}{2} \sum_I \frac{1}{r_I^2} \left[ 2P^I_{(i} n^I_{j)} - (f_{ij} - n_{iI}n_{jI})P^I_k n^I_k \right] \]
\[ + \frac{4}{r_I} n^I_{(i} \epsilon_{j)k\ell} S^k_I n^\ell_I, \]

Vacuum spacetime
Involve BH only
Approach to the black holes \((r_I \to 0)\), \(\psi\) will diverge

\[
-(\partial_x^2 + \partial_y^2 + \partial_z^2)\psi = \frac{1}{8} \hat{K}^{ij} \hat{K}_{ij} \psi^{-7}
\]

\[
\psi \equiv 1 + \sum_I \frac{m_I}{2r_I} + u,
\]

\[
-(\partial_x^2 + \partial_y^2 + \partial_z^2)u = \frac{1}{8} \hat{K}^{ij} \hat{K}_{ij} \psi^{-7}
\]

\(u\) is \(C^2\) at \(r_I = 0\)

[Brandt, Bruegman, PRL (1997)]
Based on the assumption $u = \frac{a}{r}$, $a$ corresponds to the monopole of the spacetime (mass).
Weak form

\[-\nabla^2 u = f(u) \text{ in } \Omega,\]

\[\int_{\Omega} (\nabla u) \cdot (\nabla v) d^3x + \int_{\partial\Omega} \alpha uvds = \int_{\Omega} f(u)v d^3x + \int_{\partial\Omega} bvds.\]

\[u = u^i \phi_i\]

\[u^i \left[ \int_{\Omega} (\nabla \phi_i) \cdot (\nabla \phi_j) d^3x + \int_{\partial\Omega} \alpha \phi_i \phi_j ds \right]\]

\[= \int_{\Omega} f(u) \phi_j d^3x + \int_{\partial\Omega} b\phi_j ds.\]
Use Newtonian iteration method to solve above non-linear system

\[ u_{(n+1)} = u_{(n)} + \Delta u^i \phi_i \]

\[ [F'_{ij} - M_{ij} - \alpha_{ij}] \Delta u^j = \int_{\Omega} (\nabla u_{(n)}) \cdot (\nabla \phi_i) d^3x \]

\[ + \int_{\partial \Omega} \alpha u_{(n)} \phi_i ds - \int_{\Omega} f(u_{(n)}) \phi_i d^3x - \int_{\partial \Omega} b \phi_j ds \]

\[ M_{ij} = \int_{\Omega} (\nabla \phi_i) \cdot (\nabla \phi_j) d^3x, \]

\[ F'_{ij} = \int_{\Omega} \frac{df}{du} (u_{(n)}) \phi_i \phi_j d^3x, \]

\[ \alpha_{ij} = \int_{\partial \Omega} \alpha \phi_i \phi_j ds. \]
PHG and CaPHG

http://lsec.cc.ac.cn/phg/download.htm

**PHG 源码：phg-0.9.2-20150107.tar.bz2**

**相关下载：**

**所有版本 (all distributions)>>**

**集成电路寄生参数提取软件 ParAFEMImp**
ParAFEMImp is a parallel package for wideband impedance extraction in very complicated geometries of conductor. It is based on the recently developed adaptive finite element method for the circuit/field couplings problems and the development of PHG. ParAFEMImp has the potential of good parallel scalability.

**三维并行结构分析软件 PHG-Solid**
PHG-Solid is an open source parallel adaptive FEM software for 3D structural analysis. It is based on the 3D parallel adaptive finite element toolbox PHG. It features parallel adaptive finite element analysis for pure 3D structures.

**CaPHG**
The goal of CaPHG is to integrate Cactus with PHG, an FEM library to help the application development with both FEM and DG methods on unstructure meshes. The motivation is to take advantage of the programmability of the Cactus computational framework and the performance and scalability of the PHG library to provide an integrated problem solving environment to solve large scale scientific problems on high-end computing facilities.
Cactus on Unstructured Meshes with CaPHG

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Initial mesh grid generating 1

Simplest decomposition of a box:
8 vertices and 5 elements
Initial mesh grid generating 2

Using NetGen to generate mesh grid

More uniform than previous decomposition
Initial mesh grid generating 3

Putting the puncture points on vertices

Divide the computational domain into 12 subdomains
Then use NetGen to generate mesh grid

The function is $C^\infty$ inside of all elements
Uniform refinement

In order to avoid the pre-mature of the iteration, we need uniform refinement about 6 times and boundary refinement 3 times to resolve the whole computational domain.
Adaptive refinement

Error indicator: \[ E_i \equiv \sqrt{\int_{\text{i-th element}} (H^2 + \nabla H \cdot \nabla H) \, dx^3}. \]

Or: \[ E_i \equiv \sqrt{\int_{\text{i-th element}} H^2 \, dx^3}. \]

These two kind of norms result in roughly the same result in all of our tests.
Adaptive refinement

Puncture regions can be captured automatically
Boundary effect

It is harder to generate efficient grid for spherical boundary than for box.
Boundary effect

As expected, the boundary is farther, the numerical solutions is better.
Boundary effect

As expected, the boundary is farther, the numerical solutions is better.

The convergence behaves like \( \frac{1}{r_{BD}} \).

This confirms our assumption of Robin BD form.
Boundary effect

The spherical configuration is less accurate than the box configuration, so the numerical error is larger than that of box.
No matter we put the puncture points on vertexes or not, the results are roughly the same. These plots correspond to the result we do not put the puncture points on vertexes.
Polynomial order effect

Resolve the whole domain
Polynomial order effect

Resolve the puncture points region
Polynomial order effect

Real converge process
The numerical error indicator, also the Hamiltonian constraint violation. Converging behavior near puncture points.
Polynomial order effect

Higher order polynomial results in more accurate solution, at the meantime the needed resolution is coarser.
Polynomial order effect

For the usage of finite difference code AMSS-NCKU.

In practice, P1 is enough.
Effect of puncture point

Requirement for the resolution is lower, while the constraint violation is smaller

The resulted solution difference is smaller than that of polynomial order
Comparison to spectral method

The relative difference is almost uniform among the space. If the reason is resolution, we expect the spectral method is more accurate, but AFEM is more practical and more flexible, no computational cost is wasted.
From practical viewpoint, we use AMSS-NCKU code to compare the solutions. Although the FEM solution does not waste resolution, spectral solution is a little bit better on constraint violation.
Summary and prospect

- AFEM has been applied to solve punctured multi black hole ID
- Mesh grid and boundary conditions affect the numerical solution strongly
- High order polynomial bases and/or spectral bases can be used together with AFEM to combine the advantage of FD and spectral method. But much more work is needed
- Application to the evolution of Einstein equations is appreciated. This may provide high accuracy and high parallel efficiency

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