Derivation of the Divergence Cleaning Equations of Motion

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I'll use Antón et al. (2006) and Penner (2011).

We start with the modified form of Maxwell's equations in covariant form with the divergence cleaning field, ψ:

$$\nabla_\mu \left({}^*F^{\mu\nu} + g^{\mu\nu}\psi \right) = -\kappa n^\nu \psi \tag{1}$$

We can simplify the original part of Maxwell's equations:

$$\begin{aligned} \nabla_\mu &= \frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g}\, {}^*F^{\mu\nu}\right) + \Gamma^\nu{}_{\mu\lambda}\, {}^*F^{\mu\nu} &(2)\\ &= \frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g}\, {}^*F^{\mu\nu}\right) &(3) \end{aligned}$$

where the connection term vanished because ${}^*F^{\mu\nu}$ is anti-symmetric while $\Gamma^\nu{}_{\mu\lambda}$ is symmetric under permutation of its lower indices.

Using equation (18) from Antón et al. (2006),

$${}^*F^{\mu\nu} = \frac{1}{W}\left(u^\mu B^\nu - u^\nu B^\mu\right) \tag{4}$$

where B^μ is a purely spatial vector and is the magnetic field w.r.t. the hypersurfaces normal observer, i.e. the one we want. Therefore, Maxwell's equations become (remembering that $B^t = 0$ and ${}^*F^{tt} = 0$):

$$\begin{aligned} \nabla_\mu\, {}^*F^{\mu j} &= \frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g}\, {}^*F^{\mu j}\right) &(5)\\ &= \frac{1}{\sqrt{-g}}\left[\partial_t \sqrt{-g}\frac{1}{W}\left(u^t B^j\right) + \partial_i \sqrt{-g}\frac{1}{W}\left(u^i B^j - u^j B^i\right)\right] &(6)\\ &= \frac{1}{\alpha\sqrt{\gamma}}\left[\partial_t \alpha\sqrt{\gamma}\frac{1}{\alpha u^t}\left(u^t B^j\right) + \partial_i \alpha\sqrt{\gamma}\frac{1}{\alpha u^t}\left(u^i B^j - u^j B^i\right)\right] &(7)\\ &= \frac{1}{\alpha\sqrt{\gamma}}\left\{\partial_t \sqrt{\gamma} B^j + \partial_i \sqrt{\gamma}\left[\left(\alpha v^i - \beta^i\right) B^j - \left(\alpha v^j - \beta^j\right) B^i\right]\right\} &(8) \end{aligned}$$

which is equation (20) from Antón et al. (2006) divided by α. Note we have used the following identity in arriving at the last expression:

$$\frac{u^i}{u^t} = \alpha v^i - \beta^i \tag{9}$$

The divergence constraint comes from the time component of the Maxwell's equation:

$$\nabla_\mu\, {}^*F^{\mu t} = \frac{1}{\sqrt{-g}}\partial_i \left(\sqrt{-g}\, {}^*F^{it}\right) \tag{10}$$

$$= \frac{1}{\sqrt{-g}}\partial_i\sqrt{-g}\frac{1}{W}\left(u^iB^t - u^tB^i\right) \tag{11}$$

$$= -\frac{1}{\alpha\sqrt{\gamma}}\partial_i\alpha\sqrt{\gamma}\frac{1}{\alpha u^t}u^tB^i \tag{12}$$

$$= -\frac{1}{\alpha\sqrt{\gamma}}\partial_i\sqrt{\gamma}B^i \tag{13}$$

In 3+1 form, the metric's inverse is defined as

$$g^{\mu\nu} = \begin{bmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^j}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{bmatrix} \quad . \tag{14}$$

and

$$n_\mu = [-\alpha, 0, 0, 0] \quad , \quad n^\mu = \frac{1}{\alpha}\left[1, -\beta^j\right]^T \quad , \tag{15}$$

Another way to simplify/expand the divergence cleaning terms is by taking out the metric from the covariant derivative:

$$\nabla_\mu g^{\mu\nu}\psi = g^{\mu\nu}\nabla_\mu\psi = g^{\mu\nu}\partial_\mu\psi \tag{16}$$

The t-component is

$$\nabla_\mu g^{\mu t}\psi = g^{\mu t}\partial_\mu\psi = \frac{1}{\alpha^2}\left[-\partial_t\psi + \beta^i\partial_i\psi\right] \tag{17}$$

which leads to the ultimate equation:

$$\frac{1}{\alpha^2}\left[\partial_t\psi - \beta^i\partial_i\psi\right] + \frac{1}{\alpha\sqrt{\gamma}}\partial_i\sqrt{\gamma}B^i = -\frac{\kappa}{\alpha}\psi \tag{18}$$

$$\rightarrow \partial_t\psi - \beta^i\partial_i\psi + \frac{\alpha}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}B^i = -\kappa\alpha\psi \tag{19}$$

$$\rightarrow \partial_t\psi - \partial_i\left(\psi\beta^i\right) + \frac{\alpha}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}B^i = -\kappa\alpha\psi - \psi\partial_i\beta^i \tag{20}$$

$$\rightarrow \partial_t\psi - \frac{\alpha}{\sqrt{\gamma}}\frac{\sqrt{\gamma}}{\alpha}\partial_i\left(\psi\beta^i\right) + \frac{\alpha}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}B^i = -\kappa\alpha\psi - \psi\partial_i\beta^i \tag{21}$$

$$\rightarrow \partial_t\psi - \frac{\alpha}{\sqrt{\gamma}}\partial_i\left(\frac{\sqrt{\gamma}}{\alpha}\psi\beta^i\right) + \frac{\alpha}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}B^i = -\kappa\alpha\psi - \psi\partial_i\beta^i - \frac{\alpha}{\sqrt{\gamma}}\psi\beta^i\partial_i\left(\frac{\sqrt{\gamma}}{\alpha}\right) \tag{22}$$

$$\rightarrow \partial_t\psi + \frac{\alpha}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}\left(B^i - \psi\frac{\beta^i}{\alpha}\right) = -\kappa\alpha\psi - \psi\partial_i\beta^i - \frac{\alpha}{\sqrt{\gamma}}\psi\beta^i\partial_i\left(\frac{\sqrt{\gamma}}{\alpha}\right) \tag{23}$$

$$\rightarrow \partial_t\psi + \frac{\alpha}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}\left(B^i - \psi\frac{\beta^i}{\alpha}\right) = -\kappa\alpha\psi - \frac{\alpha}{\sqrt{\gamma}}\psi\partial_i\left(\frac{\sqrt{\gamma}}{\alpha}\beta^i\right) \tag{24}$$

The j-component is

$$\nabla_\mu g^{\mu j}\psi = g^{\mu j}\partial_\mu\psi = \frac{1}{\alpha^2}\left[\beta^j\partial_t\psi + \left(\alpha^2\gamma^{ij} - \beta^i\beta^j\right)\partial_i\psi\right] = \frac{\kappa}{\alpha}\beta^j\psi \tag{25}$$

$$\rightarrow \beta^j \partial_t \psi + \left(\alpha^2 \gamma^{ij} - \beta^i \beta^j\right) \partial_i \psi = \kappa \alpha \beta^j \psi \tag{26}$$

The complete $j^{\rm th}$-component of the modified Maxwell's equation becomes

$$\frac{1}{\alpha\sqrt{\gamma}} \left[\partial_t \sqrt{\gamma} B^j + \partial_i \sqrt{\gamma} \left(u^i B^j - u^j B^i\right)\right] + g^{j\mu} \partial_\mu \psi = \kappa \frac{\beta^j}{\alpha} \psi \tag{27}$$

Inserting equation 24 :

$$\frac{1}{\alpha\sqrt{\gamma}} \left[\partial_t \sqrt{\gamma} B^j + \partial_i \sqrt{\gamma} \left(u^i B^j - u^j B^i\right)\right] - \frac{\beta^j}{\alpha^2} \frac{\alpha}{\sqrt{\gamma}} \partial_i \sqrt{\gamma} \left(B^i - \psi \frac{\beta^i}{\alpha}\right) + g^{ij} \partial_i \psi \tag{28}$$

$$= \kappa \frac{\beta^j}{\alpha} \psi + \frac{\beta^j}{\alpha^2} \left\{ \kappa \alpha \psi + \frac{\alpha}{\sqrt{\gamma}} \psi \partial_i \left(\frac{\sqrt{\gamma}}{\alpha} \beta^i\right)\right\} \tag{29}$$

Simplifying both sides

$$\rightarrow \quad \frac{1}{\alpha\sqrt{\gamma}} \left[\partial_t \sqrt{\gamma} B^j + \partial_i \sqrt{\gamma} \left(u^i B^j - u^j B^i\right)\right] - \frac{1}{\alpha\sqrt{\gamma}} \beta^j \partial_i \sqrt{\gamma} \left(B^i - \psi \frac{\beta^i}{\alpha}\right) + g^{ij} \partial_i \psi \tag{30}$$

$$= 2\kappa \psi \frac{\beta^j}{\alpha} + \frac{1}{\alpha\sqrt{\gamma}} \psi \beta^j \partial_i \left(\frac{\sqrt{\gamma}}{\alpha} \beta^i\right) \tag{31}$$

Making the new terms to look like the induction equation spatial derivatives:

$$\rightarrow \quad \frac{1}{\alpha\sqrt{\gamma}} \left[\partial_t \sqrt{\gamma} B^j + \partial_i \sqrt{\gamma} \left(u^i B^j - u^j B^i\right)\right] - \frac{1}{\alpha\sqrt{\gamma}} \partial_i \sqrt{\gamma} \beta^j \left(B^i - \psi \frac{\beta^i}{\alpha}\right) + g^{ij} \partial_i \psi \tag{32}$$

$$= 2\kappa \psi \frac{\beta^j}{\alpha} + \frac{1}{\alpha\sqrt{\gamma}} \psi \beta^j \partial_i \left(\frac{\sqrt{\gamma}}{\alpha} \beta^i\right) - \frac{1}{\alpha\sqrt{\gamma}} \sqrt{\gamma} \left(B^i - \psi \frac{\beta^i}{\alpha}\right) \partial_i \beta^j \tag{33}$$

Now, combining the terms into the induction equation's spatial derivatives and expanding out the RHS:

$$\rightarrow \quad \frac{1}{\alpha\sqrt{\gamma}} \left\{ \partial_t \sqrt{\gamma} B^j + \partial_i \sqrt{\gamma} \left[u^i B^j - u^j B^i + \beta^j \left(\psi \frac{\beta^i}{\alpha} - B^i\right)\right]\right\} + g^{ij} \partial_i \psi \tag{34}$$

$$= 2\kappa \psi \frac{\beta^j}{\alpha} + \frac{1}{\alpha\sqrt{\gamma}} \psi \beta^j \left\{ \frac{\sqrt{\gamma}}{\alpha} \partial_i \beta^i + \beta^i \partial_i \frac{\sqrt{\gamma}}{\alpha}\right\} - \frac{1}{\alpha} B^i \partial_i \beta^j + \psi \frac{\beta^i}{\alpha^2} \partial_i \beta^j \tag{35}$$

Collecting terms on RHS:

$$\rightarrow \quad \frac{1}{\alpha\sqrt{\gamma}} \left\{ \partial_t \sqrt{\gamma} B^j + \partial_i \sqrt{\gamma} \left[u^i B^j - u^j B^i + \beta^j \left(\psi \frac{\beta^i}{\alpha} - B^i\right)\right]\right\} + g^{ij} \partial_i \psi \tag{36}$$

$$= 2\kappa \psi \frac{\beta^j}{\alpha} + \psi \frac{1}{\alpha^2} \partial_i \left(\beta^i \beta^j\right) + \psi \frac{\beta^i \beta^j}{\alpha\sqrt{\gamma}} \partial_i \frac{\sqrt{\gamma}}{\alpha} - \frac{1}{\alpha} B^i \partial_i \beta^j \tag{37}$$

Further collection of terms on RHS:

$$\rightarrow \quad \frac{1}{\alpha\sqrt{\gamma}} \left\{ \partial_t \sqrt{\gamma} B^j + \partial_i \sqrt{\gamma} \left[u^i B^j - u^j B^i + \beta^j \left(\psi \frac{\beta^i}{\alpha} - B^i\right)\right]\right\} + g^{ij} \partial_i \psi \tag{38}$$

$$= 2\kappa \psi \frac{\beta^j}{\alpha} + \frac{\psi}{\alpha\sqrt{\gamma}} \partial_i \left(\frac{\sqrt{\gamma} \beta^i \beta^j}{\alpha}\right) - \frac{1}{\alpha} B^i \partial_i \beta^j \tag{39}$$

Now working on the

$$\rightarrow \qquad \frac{1}{\alpha\sqrt{\gamma}}\left\{\partial_t\sqrt{\gamma}B^j + \partial_i\sqrt{\gamma}\left[u^iB^j - u^jB^i + \beta^j\left(\psi\frac{\beta^i}{\alpha} - B^i\right)\right]\right\} + \partial_i g^{ij}\psi \tag{40}$$

$$= \quad 2\kappa\psi\frac{\beta^j}{\alpha} + \frac{\psi}{\alpha\sqrt{\gamma}}\partial_i\left(\frac{\sqrt{\gamma}\beta^i\beta^j}{\alpha}\right) - \frac{1}{\alpha}B^i\partial_i\beta^j + \psi\partial_i g^{ij} \tag{41}$$

Making the spatial divergence on ψ look like the induction equation part:

$$\rightarrow \qquad \frac{1}{\alpha\sqrt{\gamma}}\left\{\partial_t\sqrt{\gamma}B^j + \partial_i\sqrt{\gamma}\left[u^iB^j - u^jB^i + \beta^j\left(\psi\frac{\beta^i}{\alpha} - B^i\right)\right]\right\} + \frac{1}{\alpha\sqrt{\gamma}}\partial_i\left(\alpha\sqrt{\gamma}g^{ij}\psi\right) \tag{42}$$

$$= \quad 2\kappa\psi\frac{\beta^j}{\alpha} + \frac{\psi}{\alpha\sqrt{\gamma}}\partial_i\left(\frac{\sqrt{\gamma}\beta^i\beta^j}{\alpha}\right) - \frac{1}{\alpha}B^i\partial_i\beta^j + \psi\partial_i g^{ij} + \frac{g^{ij}\psi}{\alpha\sqrt{\gamma}}\partial_i\left(\alpha\sqrt{\gamma}\right) \tag{43}$$

Absorb the new derivatives into the induction equation part:

$$\rightarrow \qquad \frac{1}{\alpha\sqrt{\gamma}}\left\{\partial_t\sqrt{\gamma}B^j + \partial_i\sqrt{\gamma}\left[u^iB^j - u^jB^i + \alpha g^{ij}\psi + \beta^j\left(\psi\frac{\beta^i}{\alpha} - B^i\right)\right]\right\} \tag{44}$$

$$= \quad 2\kappa\psi\frac{\beta^j}{\alpha} + \frac{\psi}{\alpha\sqrt{\gamma}}\partial_i\left(\frac{\sqrt{\gamma}\beta^i\beta^j}{\alpha}\right) - \frac{1}{\alpha}B^i\partial_i\beta^j + \psi\partial_i g^{ij} + \frac{g^{ij}\psi}{\alpha\sqrt{\gamma}}\partial_i\left(\alpha\sqrt{\gamma}\right) \tag{45}$$

The "absorption" cancels terms when using 3+1 form:

$$\rightarrow \qquad \frac{1}{\alpha\sqrt{\gamma}}\left\{\partial_t\sqrt{\gamma}B^j + \partial_i\sqrt{\gamma}\left[u^iB^j - u^jB^i + \alpha\gamma^{ij}\psi - \beta^jB^i\right]\right\} \tag{46}$$

$$= \quad 2\kappa\psi\frac{\beta^j}{\alpha} + \frac{\psi}{\alpha\sqrt{\gamma}}\partial_i\left(\frac{\sqrt{\gamma}\beta^i\beta^j}{\alpha}\right) - \frac{1}{\alpha}B^i\partial_i\beta^j + \psi\partial_i g^{ij} + \frac{g^{ij}\psi}{\alpha\sqrt{\gamma}}\partial_i\left(\alpha\sqrt{\gamma}\right) \tag{47}$$

Collecting last two terms on RHS:

$$\rightarrow \qquad \frac{1}{\alpha\sqrt{\gamma}}\left\{\partial_t\sqrt{\gamma}B^j + \partial_i\sqrt{\gamma}\left[u^iB^j - u^jB^i + \alpha\gamma^{ij}\psi - \beta^jB^i\right]\right\} \tag{48}$$

$$= \quad 2\kappa\psi\frac{\beta^j}{\alpha} + \frac{\psi}{\alpha\sqrt{\gamma}}\partial_i\left(\frac{\sqrt{\gamma}\beta^i\beta^j}{\alpha}\right) - \frac{1}{\alpha}B^i\partial_i\beta^j + \frac{\psi}{\alpha\sqrt{\gamma}}\partial_i\left(\alpha\sqrt{\gamma}g^{ij}\right) \tag{49}$$

Using 3+1 form of metric:

$$\rightarrow \qquad \frac{1}{\alpha\sqrt{\gamma}}\left\{\partial_t\sqrt{\gamma}B^j + \partial_i\sqrt{\gamma}\left[u^iB^j - u^jB^i + \alpha\gamma^{ij}\psi - \beta^jB^i\right]\right\} \tag{50}$$

$$= \quad 2\kappa\psi\frac{\beta^j}{\alpha} + \frac{\psi}{\alpha\sqrt{\gamma}}\partial_i\left(\frac{\sqrt{\gamma}\beta^i\beta^j}{\alpha}\right) - \frac{1}{\alpha}B^i\partial_i\beta^j + \frac{\psi}{\alpha\sqrt{\gamma}}\partial_i\left[\sqrt{\gamma}\left(\alpha\gamma^{ij} - \frac{\beta^i\beta^j}{\alpha}\right)\right] \tag{51}$$

Cancelation of the derivative on the shift squared term:

$$\rightarrow \qquad \frac{1}{\alpha\sqrt{\gamma}}\left\{\partial_t\sqrt{\gamma}B^j + \partial_i\sqrt{\gamma}\left[u^iB^j - u^jB^i + \alpha\gamma^{ij}\psi - \beta^jB^i\right]\right\} \tag{52}$$

$$= \quad 2\kappa\psi\frac{\beta^j}{\alpha} - \frac{1}{\alpha}B^i\partial_i\beta^j + \frac{\psi}{\alpha\sqrt{\gamma}}\partial_i\left(\alpha\sqrt{\gamma}\gamma^{ij}\right) \tag{53}$$

So the new equations are :

$$\partial_t\sqrt{\gamma}B^j + \partial_i\sqrt{\gamma}\left[u^iB^j - u^jB^i + \alpha\gamma^{ij}\psi - \beta^jB^i\right] = \sqrt{\gamma}\left(2\kappa\psi\beta^j - B^i\partial_i\beta^j\right) + \psi\partial_i\left(\alpha\sqrt{\gamma}\gamma^{ij}\right) \quad (54)$$

Let us put the evolution for ψ into conservation law form. From the original derivation:

$$\rightarrow \partial_t\psi - \partial_i\left(\psi\beta^i\right) + \frac{\alpha}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}B^i = -\kappa\alpha\psi - \psi\partial_i\beta^i \quad (55)$$

$$\rightarrow \partial_t\psi - \partial_i\left(\psi\beta^i\right) + \alpha\partial_iB^i = -\kappa\alpha\psi - \psi\partial_i\beta^i - \frac{\alpha}{\sqrt{\gamma}}B^i\partial_i\sqrt{\gamma} \quad (56)$$

$$\rightarrow \partial_t\psi + \partial_i\left(\alpha B^i - \psi\beta^i\right) = -\psi\left(\kappa\alpha + \partial_i\beta^i\right) + \sqrt{\gamma}B^i\partial_i\left(\frac{\alpha}{\sqrt{\gamma}}\right) \quad (57)$$

So, the final equations are:

$$\boxed{\begin{aligned}\partial_t\sqrt{\gamma}B^j \quad & +\partial_i\sqrt{\gamma}\left[u^iB^j - u^jB^i + \alpha\gamma^{ij}\psi - \beta^jB^i\right] \\ & = \sqrt{\gamma}\left(2\kappa\psi\beta^j - B^i\partial_i\beta^j\right) + \psi\partial_i\left(\alpha\sqrt{\gamma}\gamma^{ij}\right)\end{aligned}} \quad (58)$$

$$\boxed{\partial_t\psi + \partial_i\left(\alpha B^i - \psi\beta^i\right) = -\psi\left(\kappa\alpha + \partial_i\beta^i\right) + \sqrt{\gamma}B^i\partial_i\left(\frac{\alpha}{\sqrt{\gamma}}\right)} \quad (59)$$

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