Derivation of the Divergence Cleaning Equations of Motion

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I'll use Antón et al. (2006) and Penner (2011).

We start with the modified form of Maxwell's equations in covariant form with the divergence cleaning field, ψ:

$$\nabla_\mu \left({}^*F^{\mu\nu} + g^{\mu\nu}\psi \right) = \kappa n^\nu \psi \tag{1}$$

which comes from Penner (2011) except we correct the sign of the RHS. Also, note that when $\kappa > 0$, $\nabla^\mu \nabla_\mu \psi = \kappa n^\mu \psi$ is a damped wave equation. We will thus use $\kappa > 0$ as $\partial_t \psi$ is proportional to the divergence of the magnetic field which we wish to drive to zero. We can simplify the original part of Maxwell's equations:

$$\begin{aligned}
\nabla_\mu {}^*F^{\mu\nu} &= \partial_\mu {}^*F^{\mu\nu} + \Gamma^\mu{}_{\lambda\mu} {}^*F^{\lambda\nu} + \Gamma^\nu{}_{\lambda\mu} {}^*F^{\mu\lambda} && (2)\\
&= \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, {}^*F^{\mu\nu} \right) + \Gamma^\nu{}_{\mu\lambda} {}^*F^{\mu\lambda} && (3)\\
&= \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, {}^*F^{\mu\nu} \right) && (4)
\end{aligned}$$

where the connection term vanished because ${}^*F^{\mu\lambda}$ is anti-symmetric while $\Gamma^\nu{}_{\mu\lambda}$ is symmetric under permutation of its lower indices.

Using equation (18) from Antón et al. (2006),

$${}^*F^{\mu\nu} = \frac{1}{W} \left(u^\mu B^\nu - u^\nu B^\mu \right) \tag{5}$$

where B^μ is a purely spatial vector and is the magnetic field w.r.t. the hypersurfaces normal observer, i.e. the one we want. Therefore, Maxwell's equations become (remembering that $B^t = 0$ and ${}^*F^{tt} = 0$):

$$\begin{aligned}
\nabla_\mu {}^*F^{\mu j} &= \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, {}^*F^{\mu j} \right) && (6)\\
&= \frac{1}{\sqrt{-g}} \left[\partial_t \sqrt{-g} \frac{1}{W} \left(u^t B^j \right) + \partial_i \sqrt{-g} \frac{1}{W} \left(u^i B^j - u^j B^i \right) \right] && (7)\\
&= \frac{1}{\alpha\sqrt{\gamma}} \left[\partial_t \alpha \sqrt{\gamma} \frac{1}{\alpha u^t} \left(u^t B^j \right) + \partial_i \alpha \sqrt{\gamma} \frac{1}{\alpha u^t} \left(u^i B^j - u^j B^i \right) \right] && (8)\\
&= \frac{1}{\alpha\sqrt{\gamma}} \left\{ \partial_t \sqrt{\gamma} B^j + \partial_i \sqrt{\gamma} \left[\left(\alpha v^i - \beta^i \right) B^j - \left(\alpha v^j - \beta^j \right) B^i \right] \right\} && (9)
\end{aligned}$$

which is equation (20) from Antón et al. (2006) divided by α. Note we have used the following identity in arriving at the last expression:

$$\frac{u^i}{u^t} = \alpha v^i - \beta^i \tag{10}$$

The divergence constraint comes from the time component of the Maxwell's equation:

$$
\begin{aligned}
0 = \nabla_\mu {}^*F^{\mu t} &= \frac{1}{\sqrt{-g}}\partial_i\left(\sqrt{-g}\,{}^*F^{it}\right) &(11)\\
&= \frac{1}{\sqrt{-g}}\partial_i\sqrt{-g}\frac{1}{W}\left(u^iB^t - u^tB^i\right) &(12)\\
&= -\frac{1}{\alpha\sqrt{\gamma}}\partial_i\alpha\sqrt{\gamma}\frac{1}{\alpha u^t}u^tB^i &(13)\\
&= -\frac{1}{\alpha\sqrt{\gamma}}\partial_i\sqrt{\gamma}B^i &(14)
\end{aligned}
$$

In 3+1 form, the metric's inverse is defined as

$$
g^{\mu\nu} = \begin{bmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^j}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{bmatrix} . \qquad (15)
$$

and

$$
n_\mu = [-\alpha, 0, 0, 0] \quad , \quad n^\mu = \frac{1}{\alpha}\left[1, -\beta^j\right]^T \quad , \qquad (16)
$$

To simplify/expand the divergence cleaning term in Eq. 1, we may take out the metric from its covariant derivative:

$$
\nabla_\mu g^{\mu\nu}\psi = g^{\mu\nu}\nabla_\mu\psi = g^{\mu\nu}\partial_\mu\psi \qquad (17)
$$

The t-component is

$$
\nabla_\mu g^{\mu t}\psi = g^{\mu t}\partial_\mu\psi = \frac{1}{\alpha^2}\left[-\partial_t\psi + \beta^i\partial_i\psi\right] \qquad (18)
$$

which leads to the ultimate equation:

$$
\frac{1}{\alpha^2}\left[\partial_t\psi - \beta^i\partial_i\psi\right] + \frac{1}{\alpha\sqrt{\gamma}}\partial_i\sqrt{\gamma}B^i = -\frac{\kappa}{\alpha}\psi \qquad (19)
$$

$$
\rightarrow \partial_t\psi - \beta^i\partial_i\psi + \frac{\alpha}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}B^i = -\kappa\alpha\psi \qquad (20)
$$

$$
\boxed{\partial_t\psi + \partial_i\left(\alpha B^i - \psi\beta^i\right) = \psi\left(-\kappa\alpha - \partial_i\beta^i\right) + \sqrt{\gamma}B^i\partial_i\left(\frac{\alpha}{\sqrt{\gamma}}\right)} \qquad (21)
$$

The j-component is

$$
\nabla_\mu g^{\mu j}\psi = g^{\mu j}\partial_\mu\psi = \frac{1}{\alpha^2}\left[\beta^j\partial_t\psi + \left(\alpha^2\gamma^{ij} - \beta^i\beta^j\right)\partial_i\psi\right] = -\frac{\kappa}{\alpha}\beta^j\psi \qquad (22)
$$

$$
\rightarrow \beta^j\partial_t\psi + \left(\alpha^2\gamma^{ij} - \beta^i\beta^j\right)\partial_i\psi = -\kappa\alpha\beta^j\psi \qquad (23)
$$

The complete j^{th}-component of the modified Maxwell's equation becomes

$$\frac{1}{\alpha\sqrt{\gamma}}\left[\partial_t\sqrt{\gamma}B^j + \partial_i\sqrt{\gamma}\left(u^iB^j - u^jB^i\right)\right] + g^{j\mu}\partial_\mu\psi = -\kappa\frac{\beta^j}{\alpha}\psi \tag{24}$$

Inserting equation 21 :

$$\frac{1}{\alpha\sqrt{\gamma}}\left[\partial_t\sqrt{\gamma}B^j + \partial_i\sqrt{\gamma}\left(u^iB^j - u^jB^i\right)\right] - \frac{\beta^j}{\alpha^2}\partial_i\left(\alpha B^i - \psi\beta^i\right) + g^{ij}\partial_i\psi \tag{25}$$

$$= \quad -\kappa\frac{\beta^j}{\alpha}\psi + \frac{\beta^j}{\alpha^2}\left\{\kappa\alpha\psi + \psi\partial_i\beta^i - \sqrt{\gamma}B^i\partial_i\left(\frac{\alpha}{\sqrt{\gamma}}\right)\right\} \tag{26}$$

Multiplying through by $\sqrt{-g} = \alpha\sqrt{\gamma}$:

$$\partial_t\sqrt{\gamma}B^j + \partial_i\sqrt{\gamma}\left(u^iB^j - u^jB^i\right) - \frac{\sqrt{\gamma}\beta^j}{\alpha}\partial_i\left(\alpha B^i - \psi\beta^i\right) + \alpha\sqrt{\gamma}g^{ij}\partial_i\psi \tag{27}$$

$$= \quad \frac{\sqrt{\gamma}\beta^j}{\alpha}\left\{\psi\partial_i\beta^i - \sqrt{\gamma}B^i\partial_i\left(\frac{\alpha}{\sqrt{\gamma}}\right)\right\} \tag{28}$$

Grouping spatial derivatives:

$$\partial_t\sqrt{\gamma}B^j + \partial_i\sqrt{\gamma}\left(u^iB^j - u^jB^i\right) + \frac{\sqrt{\gamma}\beta^j}{\alpha}\left[\sqrt{\gamma}B^i\partial_i\left(\frac{\alpha}{\sqrt{\gamma}}\right) - \partial_i(\alpha B^i)\right] +$$
$$\alpha\sqrt{\gamma}\left(g^{ij}\partial_i\psi + \frac{\beta^j}{\alpha^2}[\partial_i(\psi\beta^i) - \psi\partial_i\beta^i]\right) = 0 \tag{29}$$

Simplifying the terms:

$$\partial_t\sqrt{\gamma}B^j + \partial_i\sqrt{\gamma}\left(u^iB^j - u^jB^i\right) - \frac{\sqrt{\gamma}\beta^j}{\alpha}\left[\frac{\alpha}{\sqrt{\gamma}}\partial_i(\sqrt{\gamma}B^i)\right] + \alpha\sqrt{\gamma}\left(g^{ij} + \frac{\beta^i\beta^j}{\alpha^2}\right)\partial_i\psi$$
$$= 0 \tag{30}$$

which reduces to:

$$\partial_t\sqrt{\gamma}B^j + \partial_i\sqrt{\gamma}\left(u^iB^j - u^jB^i\right) - \beta^j\partial_i(\sqrt{\gamma}B^i) + \alpha\sqrt{\gamma}\gamma^{ij}\partial_i\psi = 0 \tag{31}$$

or in conservative form:

$$\boxed{\begin{aligned}\partial_t\sqrt{\gamma}B^j \quad & +\partial_i\sqrt{\gamma}\left[u^iB^j - u^jB^i + \alpha\gamma^{ij}\psi - \beta^jB^i\right] \\ & = -\sqrt{\gamma}\left(B^i\partial_i\beta^j\right) + \psi\partial_i\left(\alpha\sqrt{\gamma}\gamma^{ij}\right)\end{aligned}} \tag{32}$$

REFERENCES

J. Penner, *PhD Thesis*, University of British Columbia (2011); http://bh0.physics.ubc.ca/People/ajpenner/thesis/phd.pdf.

L. Antón, et al., *ApJ*, **637**, 296-312 (2006).

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