1 MHD Con2Prim

1.1 Notation

The current MHD con2prim routines assume a polytropic equation of state, but may be generalized to include arbitrary equations of state so long as we may evaluate the pressure as a function of density and internal energy, as well as its derivatives with respect to those quantities.

To establish notation, we follow that of Noble et al. (2006) [astro-ph/0512420], which describes the Con2Prim routines implemented in the `HARM` code, specifically, the $1D_W$ and $2D$ schemes. We define the following quantities:

$$
\begin{aligned}
\text{Rest mass density} &: \quad \rho_0 \\
\text{Internal energy} &: \quad u \\
\text{Pressure} &: \quad p \\
\text{Enthalpy} &: \quad w \equiv \rho_0 + u + p \\
3-\text{velocity} &: \quad v \quad \text{(Valencia definition)} : v_i = \gamma u_i \\
\text{Lorentz factor} &: \quad \gamma = (1 - v^2)^{-1/2}
\end{aligned}
$$

We begin from eight given conserved quantities: $D,\ S_i,\ \tau,\ B^i$, all of which have been undensitized by dividing the typical `GRHydro` forms by $\sqrt{|\gamma_{ij}|} \equiv \psi^6$.

We may define a few auxiliary quantities for use in our numerical calculations. Define the momentum density

$$Q_\mu = \alpha T^0_\mu; \quad Q_i \equiv S_i \tag{1}$$

and its normal projection $\tilde{Q}$ given by

$$\tilde{Q}_\mu = Q_\mu + (Q_\nu n^\nu) n_\mu \tag{2}$$

both of which are known from the conservative variables and the metric, as well as the quantity

$$W \equiv w\gamma^2 \tag{3}$$

1.2 2-d Newton-Raphson for General EOS

The 2-d Newton-Raphson approach implemented currently solves the following two equations for the unknown quantities W and v^2, with all other terms known from the given conserved set:

$$\tilde{Q}^2 = v^2(B^2 + W)^2 - (Q \cdot B)^2 \frac{B^2 + 2W}{W^2} \tag{4}$$

$$Q \cdot n = -\frac{B^2}{2}(1 + v^2) + \frac{(Q \cdot B)^2}{2} W^{-2} - W + p(\rho_0, u) \tag{5}$$

where all dot products are understood as 4-d here. The problematic step is found in the routine `eos_info`, where we actually calculate p from the conserved variables and the current guess for W and v^2 by noting that

$$\begin{aligned} (1-v^2)W &= \frac{W}{\gamma^2} = w = \rho_0 + u + p \\ \sqrt{1-v^2}\ D &= \frac{D}{\gamma} = \rho_0 \\ u + p &= (1-v^2)W - \sqrt{1-v^2}\ D \end{aligned} \tag{6}$$

and then note that for a polytrope that $p = (\Gamma - 1)u$.

For a more general EOS, it is easier given the current structure of the EOS interface in the EinsteinToolkit to solve for u, or, equivalently, $\epsilon = u/\rho_0$, and then use it in the 2-D NR steps. First, noting that

$$p = w - \rho_0 - u = \frac{W}{\gamma^2} - \rho_0 - u = (1-v^2)W - D\sqrt{1-v^2} - u \tag{7}$$

we may rewrite Eq. 5 as

$$Q \cdot n = -\frac{B^2}{2}(1+v^2) + \frac{(Q\cdot B)^2}{2}W^{-2} - v^2 W - D\sqrt{1-v^2} - u \tag{8}$$

To perform a NR step, we first set $\rho_0 = D\sqrt{1-v^2}$ and solve

$$u + p(\rho_0, u) = W(1-v^2) - D\sqrt{1-v^2} \tag{9}$$

which should work smoothly since the left hand side will in general be a monotonic function of u. Next, we use those values and calculate the derivatives. We find for $\partial u/\partial W$:

$$\begin{aligned} \frac{\partial u}{\partial W} + \left(\frac{\partial p}{\partial u}\right)_{\rho_0} \frac{\partial u}{\partial W} &= 1 - v^2 \\ \frac{\partial u}{\partial W} &= \frac{1-v^2}{1 + \left(\frac{\partial p}{\partial u}\right)_{\rho_0}} \end{aligned} \tag{10}$$

and for $\partial u/\partial(v^2)$:

$$\begin{aligned} \frac{\partial u}{\partial(v^2)} + \left(\frac{\partial p}{\partial \rho_0}\right)_u \left[\frac{-D}{2\sqrt{1-v^2}}\right] + \left(\frac{\partial p}{\partial u}\right)_{\rho_0} \frac{\partial u}{\partial(v^2)} &= -W + \frac{D}{2\sqrt{1-v^2}} \\ \frac{\partial u}{\partial(v^2)} = \frac{\frac{D}{2\sqrt{1-v^2}}\left[1 + \left(\frac{\partial p}{\partial \rho_0}\right)_u\right] - W}{1 + \left(\frac{\partial p}{\partial u}\right)_{\rho_0}} & \end{aligned} \tag{11}$$

Finally, in the Newton-Raphson step, we find by taking derivatives of Eq. 8 that the corresponding Jacobian entries in `func_vsq` are given by

$$\begin{aligned} J(1,0) &: -\frac{(Q\cdot B)^2}{W^3} - v^2 - \frac{\partial u}{\partial W} \\ J(1,1) &: -\frac{B^2}{2} - W + \frac{D}{2\sqrt{1-v^2}} - \frac{\partial u}{\partial(v^2)} \end{aligned}$$

with no change required for the residuals so long as we record the corresponding value of $p(\rho_0, u)$.

Note that for an EOS defined such that $p = p(\rho_0, \epsilon)$, the derivatives above are given by

$$\begin{aligned}
\left(\frac{\partial p}{\partial \rho_0}\right)_u &= \left(\frac{\partial p}{\partial \rho_0}\right)_\epsilon - \frac{\epsilon}{\rho_0}\left(\frac{\partial p}{\partial \epsilon}\right)_{\rho_0} \\
\left(\frac{\partial p}{\partial u}\right)_{\rho_0} &= \frac{1}{\rho_0}\left(\frac{\partial p}{\partial \epsilon}\right)_{\rho_0}
\end{aligned}$$

1.2.1 An explicit version for general EOS

Note that if the loop-with-a-loop proves problematic, one may also use v^2 and ϵ as fundamental variables to construct a fully explicit scheme, and find

$$\begin{aligned}
\rho_0 &= D\sqrt{1-v^2} \\
W &= \frac{\rho_0[1+\epsilon] + p(\rho_0, \epsilon)}{1-v^2}
\end{aligned}$$

plugging the latter into Eqs. 4 and 5 and properly evaluating the Jacobian matrix. Unfortunately, while a scheme using W and ϵ might yield simpler derivatives in the Jacobian, solving for $v(W, \epsilon)$ seems like a particularly daunting task.

1.3 The 1-d Newton Raphson solver for polytype EOS

For cases where the pressure and internal energy are functions of the rest mass density only, called "polytype" throughout `GRHydro`, we need a different Con2Prim inversion technique since the quantity $Q \cdot n$ in Eqs. 5 / 8 requires knowledge of τ, which is not evolved in these cases. Instead, the inversion uses Eq. 4 only, as follows.

We may use Eq. 4 to eliminate the variable v^2 from the Newton-Raphson scheme by solving it for $v^2(W)$:

$$v^2(W) = \frac{W^2\tilde{Q}^2 + (Q \cdot B)^2(B^2 + 2W)}{w^2(B^2+W)^2} \tag{12}$$

To perform the iteration, we may proceed by first solving for $\rho_0(W)$ through an independent Newton-Raphson loop over the equation

$$\begin{aligned}
W &= w\gamma^2 = \frac{wD^2}{\rho_0^2} \\
\rho_0 W &= D^2\left(1 + \frac{\Gamma K \rho^{\Gamma-1}}{\Gamma - 1}\right) \quad \text{Polytropic} \qquad (13) \\
\rho_0 W &= D^2(1 + \epsilon(\rho_0) + p(\rho_0)) \quad \text{General} \qquad (14)
\end{aligned}$$

which requires knowledge of the two first derivatives $dp/d\rho_0$ and $d\epsilon/d\rho_0$.

Next, we use Eq. 12 and the fact that

$$v^2 = \frac{\rho_0^2}{D^2} - 1 \tag{15}$$

to replace Eq. 4 by an expression given only in terms of W and $\rho_0(W)$:

$$\begin{aligned} 0 &= W^2(B^2+W)^2v^2 - W^2(B^2+W)^2v^2 \\ &= W^2\tilde{Q}^2 + (Q\cdot B)^2(B^2+2W) - \left(\frac{\rho_0^2}{D^2} - 1\right) W^2(B^2+W)^2 \end{aligned} \tag{16}$$

To evaluate the Newton-Raphson step, all quantities in the equation above are constants except W and $\rho_0(W)$, for which the derivative of Eqs. 13 / 14 is given by

$$\begin{aligned} \rho_0 + W\frac{d\rho_0}{dW} &= D^\gamma K\rho^{\Gamma-2}\frac{d\rho_0}{dW} \\ \frac{d\rho_0}{dW} &= \frac{\rho_0}{D^2\gamma K\rho^{\Gamma-2} - W} \quad \text{Polytropic} \end{aligned} \tag{17}$$

$$\begin{aligned} \rho_0 + W\frac{d\rho_0}{dW} &= D^2\left(\frac{d\epsilon}{d\rho_0} + \frac{dp}{d\rho_0}\right)\frac{d\rho_0}{dW} \\ \frac{d\rho_0}{dW} &= \frac{\rho_0}{D^2\left(\frac{d\epsilon}{d\rho_0} + \frac{dp}{d\rho_0}\right) - W} \quad \text{General} \end{aligned} \tag{18}$$